SETS OF COMPLEX NUMBERS ASSOCIATED WITH A MATRIX

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1. **Introduction.** Let $A = (a_{ij})$ be a square matrix of order n whose elements are complex numbers. If $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ are vectors such that

$$(1) \qquad x\bar{x}' = \sum_{i=1}^{n} x_i\bar{x}_i = 1, \qquad y\bar{y}' = \sum_{i=1}^{n} y_i\bar{y}_i = 1,$$

then $xA\bar{y}' = \sum_{ij=1}^{n} a_{ij}x_i\bar{y}_j = \alpha$, where α is a complex number. If S_1 is the set of all complex numbers of the form $xA\bar{y}'$ where x and y satisfy conditions (1), then S_1 is the set of all complex numbers in or on the circle of radius ρ_n about zero in the complex plane, where ρ_n^2 is the largest of the characteristic roots of $A\bar{A}'$ (see [3]). It is the purpose of this paper to investigate this set further and also to investigate two subsets of this set. The set S_1 is the set of elements of all matrices $UA\bar{V}'$ where U and V are unitary matrices $(U\bar{U}' = V\bar{V}' = I)$.

The set S_2 consisting of all complex numbers of the form $xA\bar{x}'$, where x satisfies (1), is a closed convex set in the complex plane and is called the *field of values of A* (see [1]). The set S_2 is the set of all diagonal elements of all matrices $UA\bar{U}'$ where U is a unitary matrix. Hence S_2 is unchanged if A is replaced by $UA\bar{U}'$. The set S_3 consisting of all complex numbers of the form $xA\bar{y}'$, where x and y satisfy (1) and also $x\bar{y}' = 0$, is the set of all non-diagonal elements of all matrices $UA\bar{U}'$ where U is a unitary matrix. The set S_3 is also unchanged if A is replaced by $UA\bar{U}'$.

2. **The sets S_2 and S_3.** If the characteristic roots of $A\bar{A}'$ are $\rho_1^2 \leq \rho_2^2 \leq \cdots \leq \rho_n^2$ and $R = \text{diag.}\{\rho_1, \rho_2, \cdots, \rho_n\}$ where $\rho_i \geq 0$, there exist unitary matrices U and V such that $\bar{U}'AV = R$ (see [2; 78]). Hence $UR\bar{V}' = A = (a_{ij})$ and $a_{ij} = u_iR\bar{v}_j'$, where $u_i = (u_{i1}, u_{i2}, \cdots, u_{in})$ and $v_i = (v_{i1}, v_{i2}, \cdots, v_{in})$ are the i-th rows of U and V respectively. Write $|u_{ik}| = \xi_{ik}$ and $|v_{ik}| = \eta_{ik}$ and it follows that

$$|a_{ij}| \leq \sum_{k=1}^{n} \rho_k\xi_{ik}\eta_{jk} \leq \tfrac{1}{2}\sum_{k=1}^{n} \rho_k(\xi_{ik}^2 + \eta_{jk}^2) \leq \tfrac{1}{2}\rho_n \sum_{k=1}^{n} (\xi_{ik}^2 + \eta_{jk}^2) = \rho_n,$$

since

$$\sum_{k=1}^{n} \xi_{ik}^2 = \sum_{k=1}^{n} \eta_{jk}^2 = 1.$$

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THEOREM 1. *The elements of a matrix* A *are all in or on the circle of radius* ρ_n *about zero in the complex plane, where* ρ_n^2 *is the largest of the characteristic roots of* $A\bar{A}'$.

The matrix R is unchanged if A is replaced by any one of its unitary transforms $(UA\bar{U}')$. Hence all elements of S_2 and S_3 are in or on this circle.

THEOREM 2. *The field of values of* A *is contained in the circle of radius* ρ_n *about zero in the complex plane, where* ρ_n^2 *is the largest of the characteristic roots of* $A\bar{A}'$.

If in the vector x each component $x_i = t_i/n^{\frac{1}{2}}$ where t_i is a unit scalar, then $xA\bar{x}' = n^{-1} \sum_{ij=1}^{n} a_{ij}t_i\bar{t}_j$ and this number belongs to S_2. In particular for $t_i = t$, $i = 1, 2, \cdots, n$, $xA\bar{x}' = n^{-1} \sum_{ij=1}^{n} a_{ij}$. The field of values of A_k, any k-rowed principal sub-matrix of A, lies in S_2 and hence (see [4]) we have

THEOREM 3. *The field of values of* A *contains* s_k/k *where* s_k *is the sum of the elements in any* k*-rowed principal sub-matrix of* A *or of any of its unitary transforms for* $k = 1, 2, \cdots, n$.

If $\mu = xA\bar{y}'$, $x\bar{y}' = 0$, is in S_3 and t is a unit scalar, then $t\mu = (tx)A\bar{y}'$ and $(tx)\bar{y}' = t(x\bar{y}') = 0$ so that $t\mu$ is also in S_3. Since there is a unitary matrix U such that $UA\bar{U}'$ is triangular, the number zero is in S_3.

If $B = A - mI$ where m is a scalar, and U is a unitary matrix, then $UB\bar{U}' = UA\bar{U}' - mI$. Hence the set S_3 for the matrix B is the same as the set S_3 for the matrix A. If the characteristic roots of $B\bar{B}'$ are $\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2$ where $0 \leq \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n$, then σ_n is a function of m and the elements of the set S_3 are all in or on the circle about zero of radius min (σ_n).

THEOREM 4. *The non-diagonal elements of all unitary transforms of the matrix* A *lie in or on the circle about zero of radius* min (σ_n), *where* σ_n^2 *is the greatest of the characteristic roots of* $(A - mI)(\bar{A}' - \bar{m}I)$.

3. **Normal matrices.** In general for non-normal matrices, the field of values of A lies within the circle of radius ρ_n. However this is not always true for if

$$A = \begin{pmatrix} k & 0 & 0 \\ 0 & c & 2b \\ 0 & 0 & -c \end{pmatrix}$$

the characteristic roots of $A\bar{A}'$ are $(a + b)^2$, $(a - b)^2$, and k^2 where $a^2 = b^2 + c^2$ and hence $\rho_3 = k$ if $k \geq a + b$.

If A is normal $(A\bar{A}' = \bar{A}'A)$ the characteristic roots of A lie on the circles of radii $\rho_1, \rho_2, \cdots, \rho_n$ (see [3]). Hence for normal matrices lower bounds for the greatest characteristic root are obtained from Theorem 1 and Theorem 3.

THEOREM 5. *Every normal matrix* $A = (a_{ij})$ *has at least one characteristic root* λ *such that* $|\lambda| \geq \max |a_{ij}|$ *and such that* $|\lambda| \geq k^{-1} \max |s_k|$, *where* s_k *is the sum of the elements in any k-rowed principal sub-matrix of* A.

COROLLARY. *If* ρ_n^2 *is the largest characteristic root of* $A\bar{A}'$, *then*

$$\rho_n^2 \geq \max \left| \sum_{k=1}^{n} a_{ik}\bar{a}_{jk} \right|, \qquad \rho_n^2 \geq n^{-1} \left| \sum_{ijk=1}^{n} a_{ik}\bar{a}_{jk} \right|.$$

If the characteristic roots of the normal matrix A are $\lambda_1, \lambda_2, \cdots, \lambda_n$ where $\lambda_k = \alpha_k + i\beta_k$, the characteristic roots of $(A + \bar{A}')/2$ are $\alpha_1, \alpha_2, \cdots, \alpha_n$ and the characteristic roots of $(A - \bar{A}')/2i$ are $\beta_1, \beta_2, \cdots, \beta_n$. Hence lower bounds for the greatest real and imaginary parts of the characteristic roots are given by the following theorem.

THEOREM 6. *If* A *is a normal matrix with characteristic roots* $\lambda_k = \alpha_k + i\beta_k$, $k = 1, 2, \cdots, n$, *then*

$$\max |\alpha_k| \geq \max |\tfrac{1}{2}(a_{ij} + \bar{a}_{ji})|, \qquad \max |\beta_k| \geq \max |\tfrac{1}{2}(a_{ij} - \bar{a}_{ji})|,$$

and also

$$\max |\alpha_k| \geq \max |(s_r + \bar{s}_r)/2r|, \qquad \max |\beta_k| \geq \max |(s_r - \bar{s}_r)/2r|,$$

where s_r *is the sum of the elements in an r-rowed principal sub-matrix of* A.

For a normal matrix it follows from Theorem 4 that $\max |\lambda_k - m| \geq |a_{ij}|$, $i \neq j$, where m is the center of the smallest circle containing the characteristic roots of the matrix. If in particular A is an Hermitian matrix with characteristic roots $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$, λ_k real, then $m = \frac{1}{2}(\lambda_1 + \lambda_n)$ and $\max |\lambda_k - m| = \frac{1}{2}(\lambda_n - \lambda_1)$ and the following theorem is established.

THEOREM 7. *If* $A = (a_{ij})$ *is an Hermitian matrix with characteristic roots* $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, *then* $\frac{1}{2}(\lambda_n - \lambda_1) \geq \max |a_{ij}|$, $i \neq j$.

For the Hermitian matrix A there exists a unitary matrix U such that $UA\bar{U}' = L = \text{diag.} \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. The set S_3 for L is identical with the set S_3 for A. If $\lambda_n = \lambda_1$, L is a scalar and hence S_3 consists of the number zero only. Assume $\lambda_n > \lambda_1$ and let x and y be real unit vectors with all components zero except the first and n-th. Since $x_1^2 + x_n^2 = 1$, $y_1^2 + y_n^2 = 1$ and $x_1y_1 + x_ny_n = 0$, it follows that $(x_1y_n - x_ny_1)^2 = 1$ and hence $x_1 = \sigma y_n$ and $x_n = -\sigma y_1$ where $\sigma = \pm 1$. Then $xL\bar{y}' = \lambda_1 x_1 y_1 + \lambda_n x_n y_n = \sigma y_1 y_n(\lambda_1 - \lambda_n) = \gamma$, and $|\gamma| = |y_1 y_n| (\lambda_n - \lambda_1) \leq \frac{1}{2}(y_1^2 + y_n^2)(\lambda_n - \lambda_1) = \frac{1}{2}(\lambda_n - \lambda_1)$. In particular if $y_1 = -y_n$, $|\gamma| = \frac{1}{2}(\lambda_n - \lambda_1)$. If γ is a real number such that $|\gamma| \leq \frac{1}{2}(\lambda_n - \lambda_1)$ the equations $y_1^2 + y_n^2 = 1$ and $y_1 y_n = \sigma\gamma/(\lambda_1 - \lambda_n)$ have solutions of the form $y_1 = \pm(r \pm s)/2$, $y_n = \pm(r \mp s)/2$ where $r^2 + s^2 = 2$. If $x_1 = \sigma y_n$ and $x_n = -\sigma y_1$, then $x_1y_1 + x_ny_n = 0$ and $x_1^2 + x_n^2 = \sigma^2(y_1^2 + y_n^2) = 1$.

THEOREM 8. *The non-diagonal elements of all unitary transforms of an Hermitian matrix A consist of all complex numbers* γ *such that* $|\gamma| \leq (\lambda_n - \lambda_1)/2$ *where* λ_1 *is the least and* λ_n *is the greatest of the characteristic roots of A.*

4. **The set** S_2. Since S_2 is a closed convex set which contains the diagonal elements of A, it also contains the number $c = n^{-1}T(A)$ where $T(A) = \sum_{i=1}^{n} a_{ii}$ is the trace of A. Let u_1 be a vector such that $u_1 A \bar{u}_1' = c$ and let U be a unitary matrix whose first row is u_1, then $UA\bar{U}'$ is a matrix whose first diagonal element is c.

Suppose that V is a unitary matrix such that

$$VA\bar{V}' = \begin{pmatrix} A_0 & A_1 \\ A_2 & A_3 \end{pmatrix}$$

where A_0 is a square matrix whose k diagonal elements are all equal to c. Since $T(VA\bar{V}') = T(A)$, $T(A_3) = (n - k)c$ and there exists a unitary matrix W_3 such that $W_3 A_3 \bar{W}_3'$ has its first diagonal elements equal to c. If

$$W = \begin{pmatrix} I_k & 0 \\ 0 & W_3 \end{pmatrix},$$

then $WVA\bar{V}'\bar{W}'$ has its first $k + 1$ diagonal elements equal to c. This completes the induction for the following theorem.

THEOREM 9. *If A is a square matrix with complex elements, there exists a unitary matrix* ϕ *such that the diagonal elements of* $\phi A \bar{\phi}'$ *are all equal.*

In particular if $A = L = \text{diag.}\{\lambda_1, \lambda_2, \cdots, \lambda_n\} \neq 0$ and $\sum_{i=1}^{n} \lambda_i = 0$, there exists a unitary matrix U such that

$$\sum_{k=1}^{n} u_{ik}\bar{u}_{ik}\lambda_k = 0 \qquad (i = 1, 2, \cdots, n),$$

and hence the matrix $W = (u_{ij}\bar{u}_{ij})$ is singular. If $V = (v_{ij})$ is a unitary matrix such that $M = (v_{ij}\bar{v}_{ij})$ is singular and $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \neq 0$ is a vector such that $M\xi' = 0$ then

$$\sum_{k=1}^{n} v_{ik}\bar{v}_{ik}\xi_k = 0 \qquad (i = 1, 2, \cdots, n),$$

and hence

$$\sum_{i=1}^{n}\left(\sum_{k=1}^{n} v_{ik}\bar{v}_{ik}\xi_k\right) = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} v_{ik}\bar{v}_{ik}\right)\xi_k = \sum_{k=1}^{n} \xi_k = 0.$$

If A is a matrix with real elements and x is a real unit vector, then $xAx' = \sum_{ij=1}^{n} a_{ij}x_i x_j$ is a real number and $xAx' = xA'x' = x((A + A')/2)x'$. Since

$(A + A')/2$ is a real symmetric matrix every number in its field of values is given by a real unit vector and by argument similar to the above it follows that there exists a real orthogonal matrix ψ such that $\psi((A + A')/2)\psi'$ has its diagonal elements equal. Since $\psi A\psi'$ and $\psi A'\psi'$ have the same diagonal elements it follows that the diagonal elements of $\psi A\psi'$ are equal and hence each is $n^{-1}T(A)$.

THEOREM 10. *If A is a real matrix there exists a real orthogonal matrix ψ such that $\psi A\psi'$ has its diagonal elements all equal.*

COROLLARY 1. *A real quadratic form with matrix A is equal to k for n mutually orthogonal unit vectors if, and only if, $k = n^{-1}T(A)$.*

COROLLARY 2. *If $a = (a_1, a_2, \cdots, a_n) \neq 0$ is a real vector such that $\sum_{i=1}^n a_i = 0$, there exists a real orthogonal matrix $C = (\cos \alpha_{ij})$ such that $C_1 = (\cos^2 \alpha_{ij})$ and $C_2 = (\cos 2\alpha_{ij})$ are singular matrices and $C_1a' = C_2a' = 0$.*

If $A = \text{diag.} \{a_1, a_2, \cdots, a_n\} \neq 0$, the diagonal elements of CAC' are

$$\sum_{k=1}^n \cos^2 \alpha_{ik}a_k \qquad (i = 1, 2, \cdots, n),$$

and if each of these is $n^{-1}T(A) = 0$, $C_1a' = 0$. Since $\cos^2\alpha_{ik} = \frac{1}{2} + \frac{1}{2}\cos 2\alpha_{ik}$,

$$\sum_{k=1}^n \cos^2 \alpha_{ik}a_k = \tfrac{1}{2}\sum_{k=1}^n a_k + \tfrac{1}{2}\sum_{k=1}^n \cos 2\alpha_{ik}a_k = 0$$

and $C_2a' = 0$ since $\sum_{k=1}^n a_k = 0$.

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